The Heat Equation on the Sierpiński Gasket

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Introduction: the Sierpiński Gasket (SG)

The Sierpiński gasket (or Sierpiński triangle) is a fractal resulting from the following iterative process

Step 0. Draw an equilateral triangle T_0 of side length 1.

Step 1. $a)$ Mark the midpoints on each side of the triangle.

$b)$ Connect the three marked midpoints to form four equilateral triangles including one upside down triangle in the middle. This upside down triangle will be referred to as the middle triangle.

Step 2. Repeat Step 1 on each of the three triangles leaving the middle triangle unchanged.

Step 3. Repeat Step 2 for each of the nine equilateral triangles excluding the middle ones;

⋮

Step m. Repeat Step $m-1$;

⋮

See the first three steps in Figure 1.

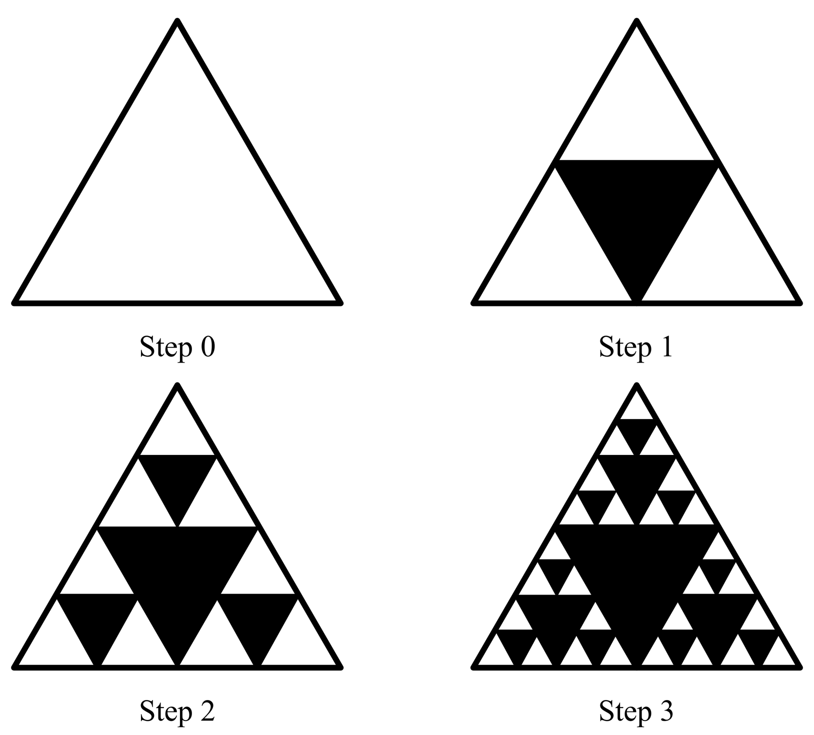

Figure 1. First steps of construction of the Sierpiński gasket

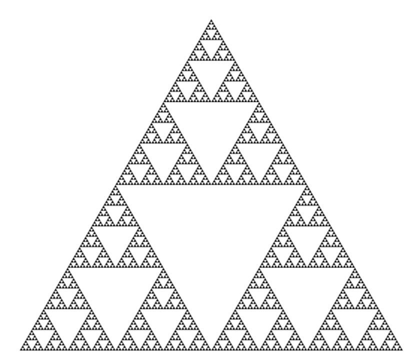

Figure 2. The Sierpiński gasket

Recursive Definition

Let T_0 be an equilateral triangle with vertices p_1, p_2, p_3. Define recursively for $n \geq 1$

$$T_n = F_1(T_{n-1}) \cup F_2(T_{n-1}) \cup F_3(T_{n-1}), \text{ where}$$

$F_1((x,y)) = \frac{1}{2}((x,y)+p_1),\ F_2((x,y)) = \frac{1}{2}((x,y)+p_2),\ F_3((x,y)) = \frac{1}{2}((x,y)+p_3)$. Then,

$$SG = \bigcap_{n=0}^{+\infty} T_n.$$

Some Geometric Properties

- At step m of the iterative process, there are 3^m triangles of side length $\frac{1}{2^m}$ and area equal to $A_m := A_0 \left(\frac{3}{4}\right)^m$, where A_0 is the area of the initial equilateral triangle. As a consequence,

$$\lim_{m \to +\infty} A_m = 0.$$

- **Its topological dimension is 1**. Nevertheless, this notion of dimension does not capture the distinctive features of the set, such as its infinite length.
- It is **self-similar** at every scale, meaning each part is a scaled copy of the whole.

The Sierpiński Gasket via Finite Graphs

A *finite graph* is a pair $G = (V, \mathcal{B})$ where:

- V is a finite set of *vertices*;
- E is a set of unordered pairs (x,y) with x, y in V and $x \neq y$, called *edges*.

A Sequence of Finite Graphs Let $V_0 = \{p_1, p_2, p_3\}$, $B_0 = \{(p_1,p_2),(p_1,p_3),(p_2,p_3)\}$

For $m \geq 1$,

$$V_m = \bigcup_{k=1}^{3} F_k(V_{m-1}),$$

where $F_k((x,y)) = \frac{1}{2}((x,y) - p_k)$ for $1 \leq k \leq 3$.

We equip the set of compact subsets $\mathbb{R}^2$ with the Hausdorff metric:

$$d_{\mathcal{H}}(X,Y) = \max\left\{\sup_{x \in X} d(x,Y), \sup_{y \in Y} d(X,y)\right\},$$

where $d(x,Y) = \inf\{d(x,y) : y \in Y\}$.

Convergence of the Sequence of Graphs to SG

Let $V_* = \bigcup_{m \geq 0} V_m$. The closure for the Hausdorff metric of V_* is the Sierpiński Gasket.

Fractals vs. Graphs - A Common Confusion

- The finite graphs $(V_m, \mathcal{B}_m)$ are just discrete approximations of the Sierpiński gasket, made of vertices and edges. They let us define analysis tools, like the Laplacian, which help us study properties on the actual gasket as we take finer approximations.
- Heat flow, eigenvalues, and other analytic properties are usually calculated on these graphs first and then shown to converge to the true fractal behavior.

Heat Equation: Continuous vs Discrete vs Fractal

Continuous case Let $u(t,x,y)$ be the temperature at time and at position (x,y) of a metallic triangle that follows the partial differential equation:

$$\begin{cases} \frac{\partial u}{\partial t} &= \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ u(0,x,y) &= \quad f(x,y), \end{cases}$$

where $f(x,y)$ is the temperature at time $t = 0$.

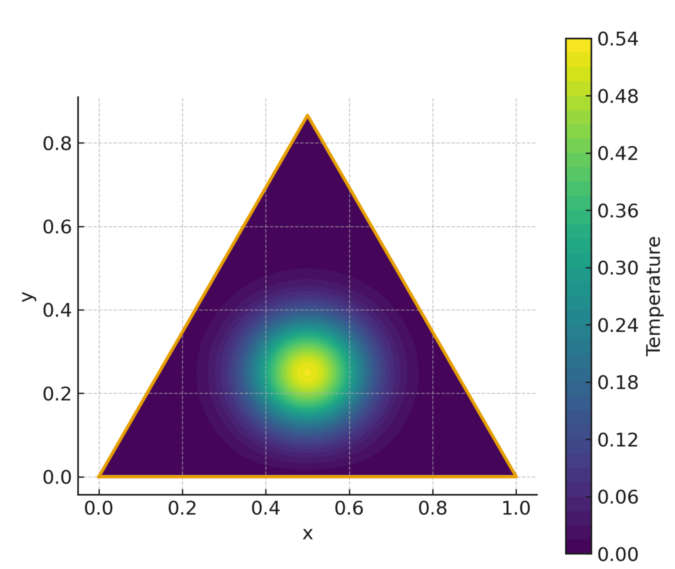

Figure 3. The Metallic Triangle

Discrete Case. Fix $m \geq 1$. Let $\ell(V_m) = \{f : f \text{ maps } V_m \text{ to } \mathbb{R}\}$. Then define a linear operator $H_m : \ell(V_m) \to \ell(V_m)$ by

$$(H_m u)(p) = \sum_{q \in V_{m,p}} (u(q) - u(p))$$

This corresponds to the discrete Laplacian on the graph.

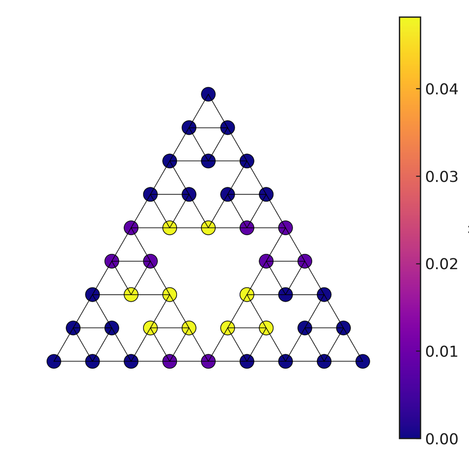

Figure 4. The Metallic Graph V_3

A Fractal Case. For $u \in C(\mathrm{SG})$ and $p \in V_m \setminus V_0$,

$$\Delta_m u(p) = \frac{3}{2} 5^m \sum_{q \in V_{m,p}} \big(u(p) - u(q)\big),$$

where the sum is over all neighbors q of p in the level-m graph. If there exists $\phi \in C(\mathrm{SG})$ such that

$$\max_{p \in V_m \setminus V_0} \left|\Delta_m u(p) - \phi(p)\right| \longrightarrow 0 \quad (m \to \infty),$$

then we set $\Delta u = \phi$ and say $u \in \mathcal{D}$.

Consider the three boundary points V_0 of the Sierpiński gasket SG as *metal contacts* held at fixed temperatures. A temperature distribution $u(t,x)$ on SG then evolves according to the heat equation

$$\partial_t u(t,p) = \Delta u(t,x,y), \qquad (x,y) \in \mathrm{SG} \setminus V_0,\ t > 0,$$

with Dirichlet boundary condition

$$u(t,p) = g(p), \qquad p \in V_0,\ t > 0,$$

and initial temperature profile

$$u(0,x) = u_0(x), \qquad x \in \mathrm{SG}.$$

Future Work

Investigate the role of edge weights: Explore how changing the conductivity of a small region in the Sierpiński gasket (by adjusting edge weights) affects the way heat spreads across the fractal using weighted Laplacian

$$(L_w u)(x) = \sum_{y \sim x} w_{xy}\,(u(x) - u(y)), \text{ for some weight } w_{xy}.$$

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References

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