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Combinatorics of Young Tableaux with Perfect Diagonals

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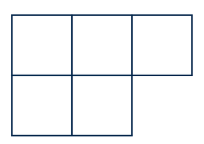
Introduction

Young Tableau: A Young tableau is a combinatorial object formed by filling the boxes of a Young diagram (a left-aligned grid of boxes with non-increasing row lengths) with positive integers.

- Rows are *weakly* increasing (left to right)
- Columns are *strictly* increasing (top to bottom)

Standard Young Tableau: A standard Young tableau is a Young tableau that:

- Fills the boxes with the numbers from 1 to n (where n is the total number of boxes)
- Entries increase *strictly* across rows and columns



Young Diagram

1	2	2
3	5	

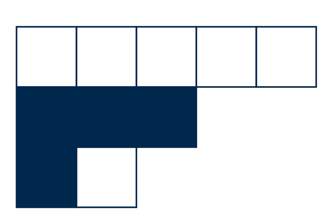
Young Tableau

1	2	4
3	5	

Standard Young Tableau

The total number of tableaux can be efficiently computed using the **Hook Length Formula**, [1]: *For a fixed shape Young diagram with n boxes, the number of standard tableaux is $n!$ divided by the product of the hook lengths of the boxes.*

Hook Length is 3



Hook Length is 4

The main goal of this research is to study the enumeration of $n \times n$ standard tableaux where the diagonals are fixed to be perfect squares by using different strategies and explore their connections to other combinatorial structures.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Standard Young Tableau

1	2	5	10
3	4	6	11
7	8	9	12
13	14	15	16

Standard Young Tableau with fixed diagonals

Observation and L-Shape

The number of possible fillings for an $n \times n$ tableau ranges from 1 to n^2.

Observe that once the diagonals are fixed, counting the possible fillings for an $n \times n$ standard tableau can be expressed with the possible combinations in the associated L-shape. It follows from the standard tableau definition that the filling range for the associated L-shape is from $(n-1)^2+1$ to n^2-1.

1	2/3
3/2	4

Tableau for $n = 2$

1	2/3	$*$
3/2	4	$*$
$*$	$*$	9

Tableau for $n = 3$

For example, after initializing the case for $n = 2$, the remaining choices to be counted for $n = 3$ are limited to the L-shape, with possible fillings being $5, 6, 7$, and 8.

This observation leads to a recursive relationship between $(n-1) \times (n-1)$ and $n \times n$ standard tableaux.

Enumeration

In order to determine the total number of standard Young tableaux under the fixed diagonal constraint, denoted by f_n, first, the admissible fillings in the **L-shape region** were manually counted for the value of n from 1 to 5. This number will be denoted by l_n.

f_n was then obtained by multiplying the total combinations for $(n-1) \times (n-1)$ with the admissible fillings of L-shape for n, establishing a recursive relation:

$$f_n = f_{n-1} \times l_n$$

n	l_n	f_n
1	1	1
2	2	2
3	6	12
4	20	240
5	70	16800

L-shape and Standard Young tableau combinations for small values of n

Python Code for Counting Specific Tableaux

```
def countValidArrangementsNxN(n):
    # Define cells array which needs to be filled
    cells = []
    for i in range(1, n):
        cells.append((i, n))
    for j in range(1, n):
        cells.append((n, j))

    numbers = list(range((n-1)**2 + 1, n**2))
    count = 0
    # Iterate over all possible orders that the numbers can be arranged
    # like (5, 6, 7, 8); (8, 7, 6, 5) for n = 3
    for perm in permutations(numbers):
        # Pairs each cell with a number from that perm
        assignment = dict(zip(cells, perm))
        assignment[(n,n)] = n**2 # Fix bottom-right corner
        valid = True
        for i in range(1, n-1):
            if (assignment[(i, n)] >= assignment[(i + 1, n)]) or
               (assignment[(n, i)] >= assignment[(n, i + 1)]):
                valid = False
                break
        if valid:
            count += 1

    return count
def youngTableauCount(n):
    if n == 1:
        return 1
    else:
        return countValidArrangementsNxN(n) * youngTableauCount(n - 1)
```

Results and Verification

After counting the L-shape tableaux and total tableaux manually and comparing them with the output from the Python program, the values matched perfectly for $n = 1$ through $n = 5$. The code also gave values for $n = 6$ as:

$$l_6 = 252 \quad \& \quad f_6 = 4{,}233{,}600.$$

Some Results

The number of admissible fillings of L-shape can be given by the **Central Binomial Coefficients**:

$$l_n = \binom{2n}{n} = \frac{(2n)!}{(n!)^2}$$

These numbers are also known as the **Type B Catalan Numbers**.

Moreover, the total number of $n \times n$ standard tableaux under the fixed diagonal constraint is determined by the **Geometric Product**:

$$f_n = \prod_{k=1}^{n} \binom{2k}{k}$$

Note: While both of these formulas are conjectured with the help of OEIS, they have not been proven yet.

Tableaux with Defects

A new notion was introduced as follows: a tableau is said to have a 1-**defect** if the entry $(n-1)^2+2$ is replaced by $(n-1)^2+1$ within the associated L-shaped region. The number of valid tableaux was subsequently recomputed to reflect this change.

1		5
	4	6
7	8	9

L-shape (no defect)

1		5
	4	5
7	8	9

L-shape with one defect at position $(n-1)^2+2$

The resulting counts were:

n	l_n = L-shape (with no defect)	L-shape (with 1-defect)
1	1	0
2	2	1
3	6	4
4	20	14
5	70	50
6	252	182

Comparison of L-shape combinations with and without 1-defect

These numbers can be expressed as a **first difference** of l_ns:

$$\#L\text{-shape with 1-defect} = l_n - l_{n-1}$$

Moreover, OEIS leads to the formula:

$$\#L\text{-shape with 1-defect} = (3n-2) \cdot C_{n-1}$$

where C_n denotes the n-th **Catalan number**.

Acknowledgment

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References

[1] William Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry*. London Mathematical Society Student Texts. Cambridge University Press, 1996.