Billiards and Distributions

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Motivation

In a game of billiards, a cue ball is struck by a cue stick (an impact). The moment of impact is instantaneous, which yields a spike. Methods from calculus cannot differentiate or integrate such spikes. We can instead model these impacts using distributions.

Billiards

In a game of billiards, the *cue ball* B is standing still until it is struck by the cue stick. That is, for a time t, $B(t)$ returns that outcome of being struck, which we will call its *impact*.

We assume that $B(0) = 0$. At the moment of impact, the ball undergoes an *instantaneous* change.

We would like to model this system and study this rate of change. Using ordinary methods from calculus, we run into immediate problems.

- The derivative is not defined at any $B(t)$. The graph of such an impact would be a spike.
- The integral of a point is 0.

Distributions

A function f belongs to $C^\infty(\mathbb{R})$ if f is differentiable at all orders: That is, $\frac{d^k f}{dx^k}$ exists and is continuous for any $k \geq 0$.

A function f is in $C_c^\infty(\mathbb{R})$ if $f \in C^\infty(\mathbb{R})$ with *compact support* where the support of f stands for the closure of $\{x \text{ in } \mathbb{R} : f(x) \neq 0\}$.

For instance, $g(x) = x^2$ is in $C^\infty(\mathbb{R})$, but does not belong to $C_c^\infty(\mathbb{R})$.
An example of a function *in* $C_c^\infty(\mathbb{R})$ is the "bump function" defined by:

$$\varphi(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

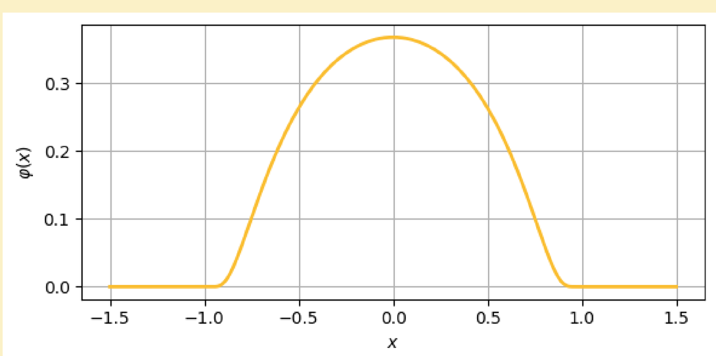


Figure 1. Graph of the bump function

We also call the set of functions that are members of $C_c^\infty(\mathbb{R})$ *test functions*.

Why They Are Called Test Functions

Suppose we have a thermometer and want to measure the temperature of a piece of heated metal. In the real world, we cannot measure the exact temperature at a single point. The tip of a thermometer is not a point; it is a bulb shape.

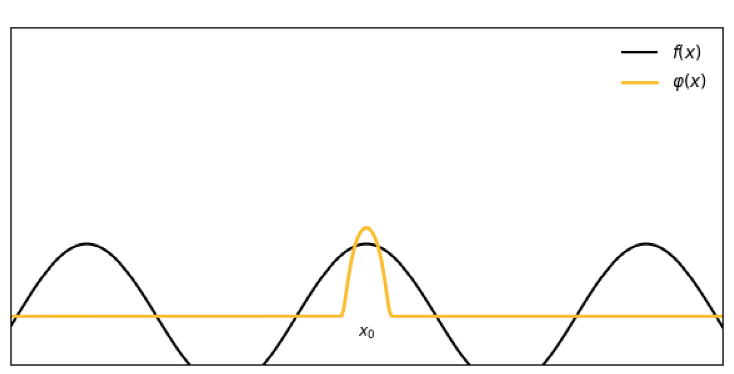
Let the bulb include two points, x_0 and x_1. Let x_0 be the point which touches the metal directly. Since it is in contact with the metal, x_0 will be the hottest point on the bulb. If x_1 is located on the opposite side of the bulb, it will be cooler.

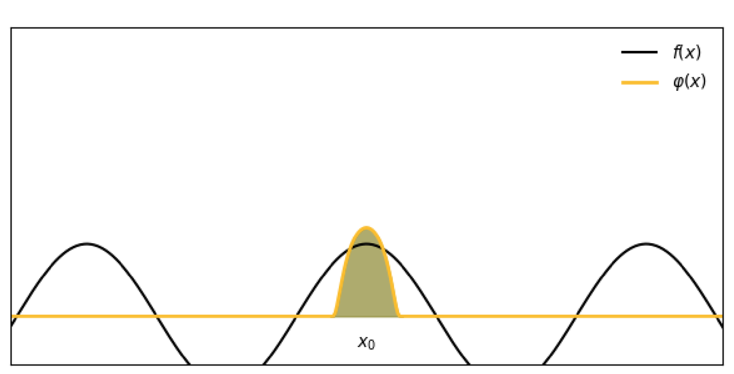
Each point on the bulb contributes differently to the final temperature. Hence, we represent these contributions as a weight function $\varphi(x)$.

We introduce the notation

$$\int f(x)\varphi(x)\,dx$$

to be a new type of integral. Here, the $f(x)$ does not have to be a true function. It is something which "acts on" test functions (which we will see shortly). Here, we have some point x_0 and $\varphi(x)$ probes a region around it. Since $\varphi(x) \in C_c^\infty(\mathbb{R})$, we get more information from the point we are interested in (the peak), and ignore information farther away as $\varphi(x)$ goes to 0.





Definition of a Distribution

A map $T : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ is called a *distribution* if:

- T is linear: $T(f + \lambda g) = Tf + \lambda Tg$, for $f, g \in C_c^\infty(\mathbb{R})$
- T is continuous: if $(f_n)_{n\geq 1}$ is a sequence of test functions such that $\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| \underset{n\to+\infty}{\longrightarrow} 0$ for some $f \in C_c^\infty(\mathbb{R})$, then,
$$|T(f_n) - T(f)| \longrightarrow 0 \quad \text{as } n \to \infty.$$

A distribution is *not* a function and it cannot in general be graphed.

Dirac δ

The Dirac distribution δ is written

$$\delta(0)[\varphi] = \int_{-\infty}^{\infty} \delta(x)\varphi(x)\,dx = \varphi(0).$$

Conversely, integration by parts gives us

$$\int_{-\infty}^{\infty} \delta'(x)\varphi(x)\,dx = -\varphi'(0).$$

The integrand $\delta(x)\varphi(x)$ is notationally convenient, ***even though*** $\delta(x)$ ***is not a function***.

We can define a *shifted* delta, centered at $t_i \in \mathbb{R}$ as

$$\delta(t - t_i)[\varphi] = \varphi(t_i).$$

We can model a sequence by summing these shifted deltas:

$$T[\varphi] = \sum_{i=1}^{n} \varphi(t_i).$$

Each term in the sum is an impact.

The Heaviside Function

The Heaviside function $H(t - t_i)$ models a sudden change at time t_i:

$$H(t - t_i) = \begin{cases} 0 & \text{if } t < t_i \\ 1 & \text{if } t \geq t_i. \end{cases}$$

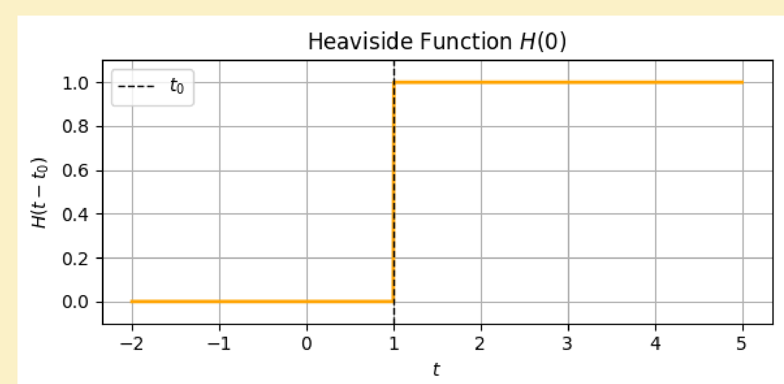
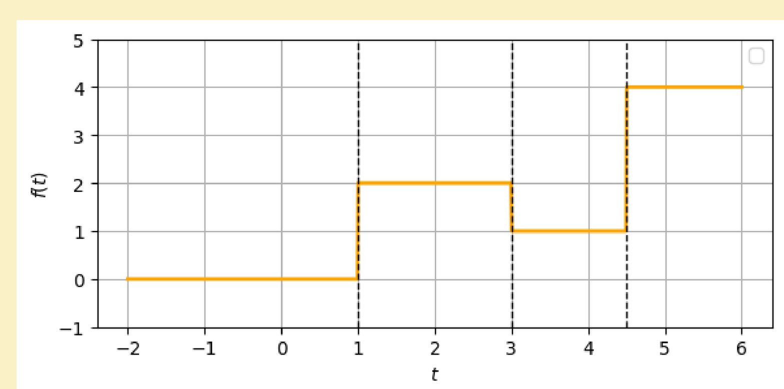


Figure 2. Heaviside Function

Each ball strike creates a new "step" in the Heaviside function. A step in our graph is an instantaneous change. The derivative of such a step is the spike centered at time t_i.



One way to define the Dirac distribution δ is as the distributional derivative of the Heaviside function H, see [2].

$$\delta[\varphi] = -\int_{-\infty}^{\infty} \varphi'(x)H(x)\,dx$$

Modeling Billiards with Distributions

Let each time the cue ball is struck be labeled t_i, and let λ_i be the magnitude of the strike at that moment.

To model the sequence of impacts over time, we use:

$$T = \sum_{i=1}^{n} \lambda_i \cdot \delta(t - t_i).$$

Each term $\lambda_i \cdot \delta(t - t_i)$ represents an instantaneous strike at time t_i, with strength λ_i.

Let $v(t) = H(t - t_i)$. Since $H'(t - t_i) = \delta(t - t_i)$, we say that acceleration is (since the derivative of a velocity function is acceleration)

$$v'(t) = a(t) = \delta(t - t_i).$$

This means that we can view the Dirac δ as the acceleration of an impact.

Two Strikes

Suppose a cue ball is struck twice: once at $t = 1$ with an impact $\lambda_1 = 2$, and again at $t = 3$ with impact $\lambda_2 = -1$. (Here, a negative λ would mean that the velocity of the ball is slowing down. For instance, if it bounces off of a wall and reverses.)

The velocity is:

$$v(t) = 2 \cdot H(t-1) - 1 \cdot H(t-3)$$

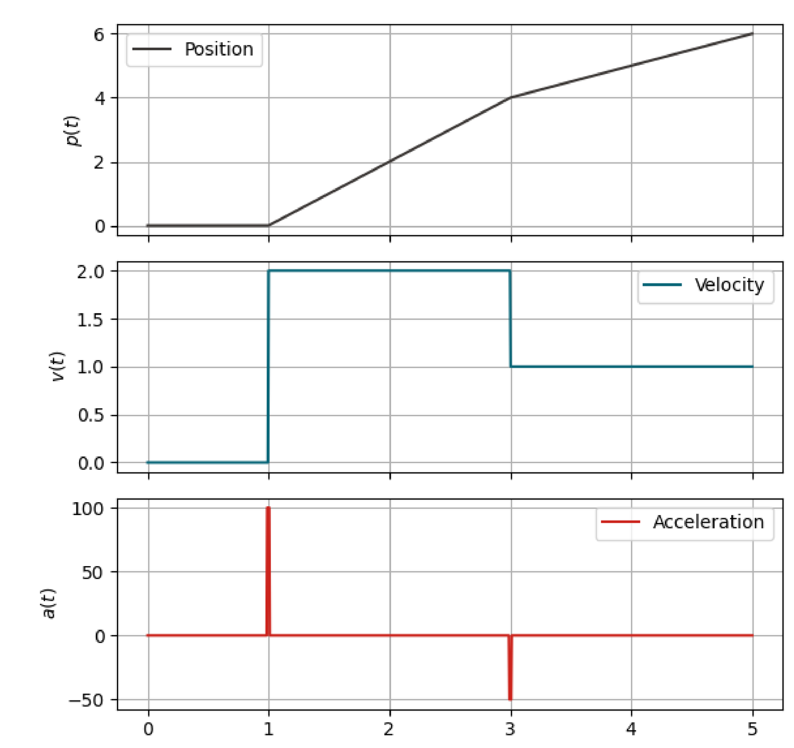
$$v(t) = \begin{cases} 0 & t < 1 \\ 2 & 1 \leq t < 3 \\ 1 & t \geq 3. \end{cases}$$

Its acceleration is:

$$v'(t) = 2 \cdot \delta(t-1) - 1 \cdot \delta(t-3).$$

And we can find the position of the ball by:

$$p(t) = \int v(t)\,dt = \begin{cases} 0 & t < 1 \\ 2(t-2) & 1 \leq t < 3 \\ t+1 & t \geq 3. \end{cases}$$



Perspective

Measuring the impact of a billiards ball impact is hard with standard calculus, but distributions make it easier. Distributional velocity and acceleration effectively capture instantaneous impacts.

We could also consider the effects of multiple balls and the dynamics arising as the cue ball strikes another, leading to a complex system involving sums of distributions.

References

[1] Axler, S. *Measure, Integration and Real Analysis*, Graduate Texts in Mathematics,282, Springer, 2020.

[2] Strauss, W. *Partial Differential Equations*, 2, John Wiley & Sons, Ltd., 2008.

[3] Strichartz, R. *A Guide to Distribution Theory and Fourier Transforms*, World Scientific Publishing Co., Inc., 2003.