Leavitt Path Algebra over Kronecker Square of Quivers

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Introduction

- Corrales Garcia et al. in [1] show that if Q_1 and Q_2 are row-finite quivers with no sinks, then there is an isomorphism between the cross product algebra $L_{\mathbb{K}}(Q_1) \otimes_G L_{\mathbb{K}}(Q_2)$ and the Leavitt path algebra $L_{\mathbb{K}}(Q_1 \times Q_2)$ where $Q_1 \times Q_2$ is not the usual product of quivers but a special kind of product. In fact, this special product $Q_1 \times Q_2$ mentioned in Corrales Garcia et al. [1] turns out to be the Kronecker product of quivers Q_1 and Q_2.
- Let $\mathcal{A}_{\mathbb{K}}(Q)$ be the quotient algebra of face algebra $\mathcal{H}_{\mathbb{K}}(\overline{Q})$ by the Cuntz-Krieger type relations. This algebra $\mathcal{A}_{\mathbb{K}}(Q)$ turns out to be isomorphic to Leavitt path algebra over a new quiver called the Kronecker square $\widehat{Q}$.

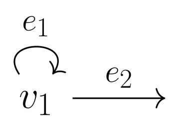
Approach

The Kronecker product of quivers was defined by Weichsel in [2]. He describes the Kronecker product of two quivers Q_1 and Q_2 as a quiver $Q_1 \otimes Q_2$ whose adjacency matrix $A_{Q_1 \otimes Q_2}$ is the Kronecker product of adjacency matrices $A_{Q_1}$and A_{Q_2}. The Kronecker product of a quiver Q with itself, $Q \otimes Q$, is called the Kronecker square of Q and denoted as $\widehat{Q}$.

Let $Q = (Q_0, Q_1, s, r)$ be a quiver. The Kronecker square $\widehat{Q} = (\widehat{Q_0}, \widehat{Q_1}, \widehat{s}, \widehat{r})$ is given by

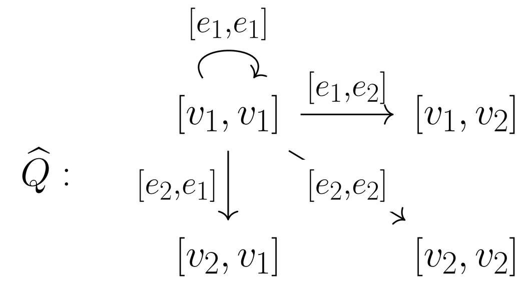
$$\begin{aligned}
\widehat{Q_0} &= \{[v_i, v_j] : v_i, v_j \in Q_0\};\\
\widehat{Q_1} &= \{[e_i, e_j] : e_i, e_j \in Q_1\};\\
\widehat{s}([e,f]) &= [s(e), s(f)];\\
\widehat{r}([e,f]) &= [r(e), r(f)].
\end{aligned}$$

Example

Let Q be a Toeplitz quiver:  whose adjacency matrix is

$$A_Q = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad A_{\widehat{Q}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, $\widehat{Q}$:



Definition

Let $Q = (Q_0, Q_1, r, s)$ be a finite quiver and its Kronecker square be $\widehat{Q} = (\widehat{Q_0}, \widehat{Q_1}, \widehat{r}, \widehat{s})$. Let K be any field. The Leavitt path algebra $L_K(\widehat{Q})$ is a K-algebra generated by $\{[v_i, v_j], [e_i, e_j], [e_i^*, e_j^*] : v_i \in Q_0, e_i \in Q_1\}$ subject to the following relations:

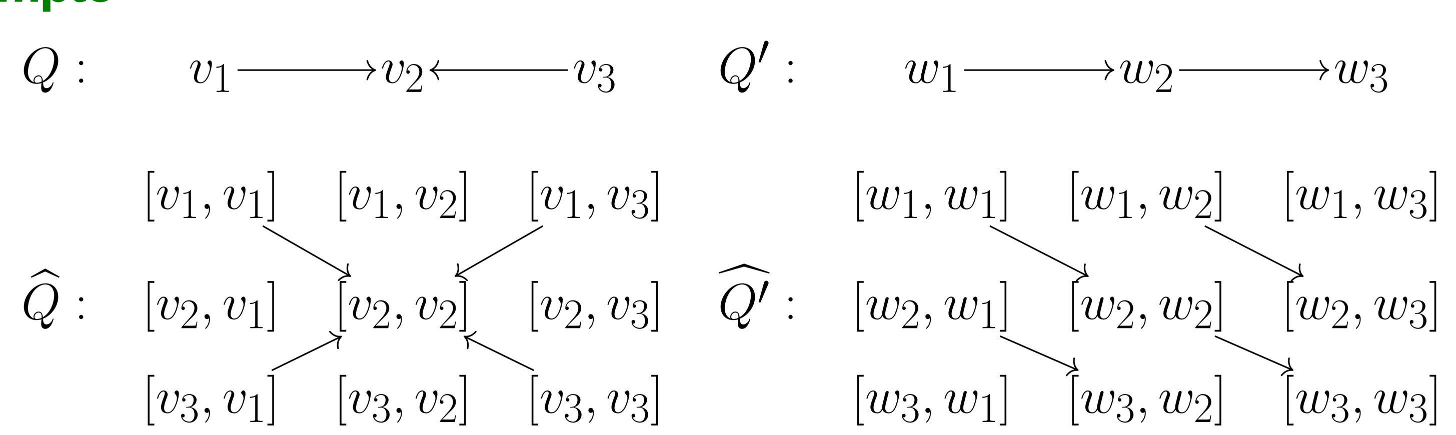
- $[v_i, v_j][v_{i'}, v_{j'}] = \delta_{i,i'}\delta_{j,j'}[v_i, v_j]$ for each $v_i, v_j, v_{i'}, v_{j'} \in Q_0$.
- $[s(e), s(f)][e,f] = [e,f]$ and $[e,f][r(e), r(f)] = [e,f]$ for each $e, f \in Q_1$.
- $[e_i^*, e_j^*][e_{i'}, e_{j'}] = \delta_{i,i'}\delta_{j,j'}[r(e_i), r(e_j)]$ for each $e_i, e_j, e_{i'}, e_{j'} \in Q_1$.
- If v_i, v_j are both regular vertices, then $[v_i, v_j] = \sum_{\substack{v_i = s(e_i),\\ v_j = s(e_j)}} [e_i, e_j][e_i^*, e_j^*]$.

Kronecker square of a quiver and its combinatorics

Let Q be a quiver and $\widehat{Q}$ its Kronecker square. Then, the following statements hold.

1. Q is finite if and only if $\widehat{Q}$ is finite. In fact, $|\widehat{Q_0}| = |Q_0|^2$ and $|\widehat{Q_1}| = |Q_1|^2$.
2. Q is acyclic if and only if $\widehat{Q}$ is acyclic.
3. A quiver Q satisfies the Condition (L) if and only if its Kronecker square $\widehat{Q}$ satisfies the Condition (L).
4. A quiver Q satisfies the Condition (K) if and only if $\widehat{Q}$ satisfies the Condition (K).
5. A quiver Q has no source or sink if and only if its Kronecker square $\widehat{Q}$ has no source or sink.
6. Let Q be a finite quiver s.t. the cycles c_i of length l_i have no exit, then $\sum_{\substack{i,j\\ i,j \geq 1}} \gcd[l_i, l_j]$ where l_i and l_j are length of cycles c_i and c_j respectively, gives the number of cycles without exit in $\widehat{Q}$.
7. A quiver Q has the countable separation property if and only if $\widehat{Q}$ has the countable separation property.
8. Let d_1, d_1' be the maximal length of a chain of cycles in Q, $\widehat{Q}$, respectively, and let d_2, d_2' be the maximal length of a chain of cycles with an exit in Q, $\widehat{Q}$, respectively. If $d_1 = k$ for a positive integer k, then
   1. $d_1' = 2k - 1$.
   2. if $d_2 < d_1$, then $d_2' = 2k - 2$, otherwise $d_2' = 2k - 1$.

Example



Ring theoretic properties of Leavitt path algebra over Kronecker square

Theorem:

Let Q be a quiver, and $\widehat{Q}$ its Kronecker square. Then

1. $L_{\mathbb{K}}(Q)$ is von Neumann regular if and only if $L_{\mathbb{K}}(\widehat{Q})$ is von Neumann regular.
2. $L_{\mathbb{K}}(Q)$ is a left/right noetherian ring if and only if $L_{\mathbb{K}}(\widehat{Q})$ is left/right noetherian.
3. $L_{\mathbb{K}}(Q)$ is a left/right artinian ring if and only if $L_{\mathbb{K}}(\widehat{Q})$ is left/right artinian.
4. $L_{\mathbb{K}}(\widehat{Q})$ is a prime (primitive) ring then $L_{\mathbb{K}}(Q)$ is a prime (primitive) ring.
5. $L_{\mathbb{K}}(Q)$ has finite polynomially bounded growth if and only if $L_{\mathbb{K}}(\widehat{Q})$ has polynomially bounded growth.
6. For a finite quiver Q, if $\mathrm{GKdim}(L_{\mathbb{K}}(Q)) = n$, then

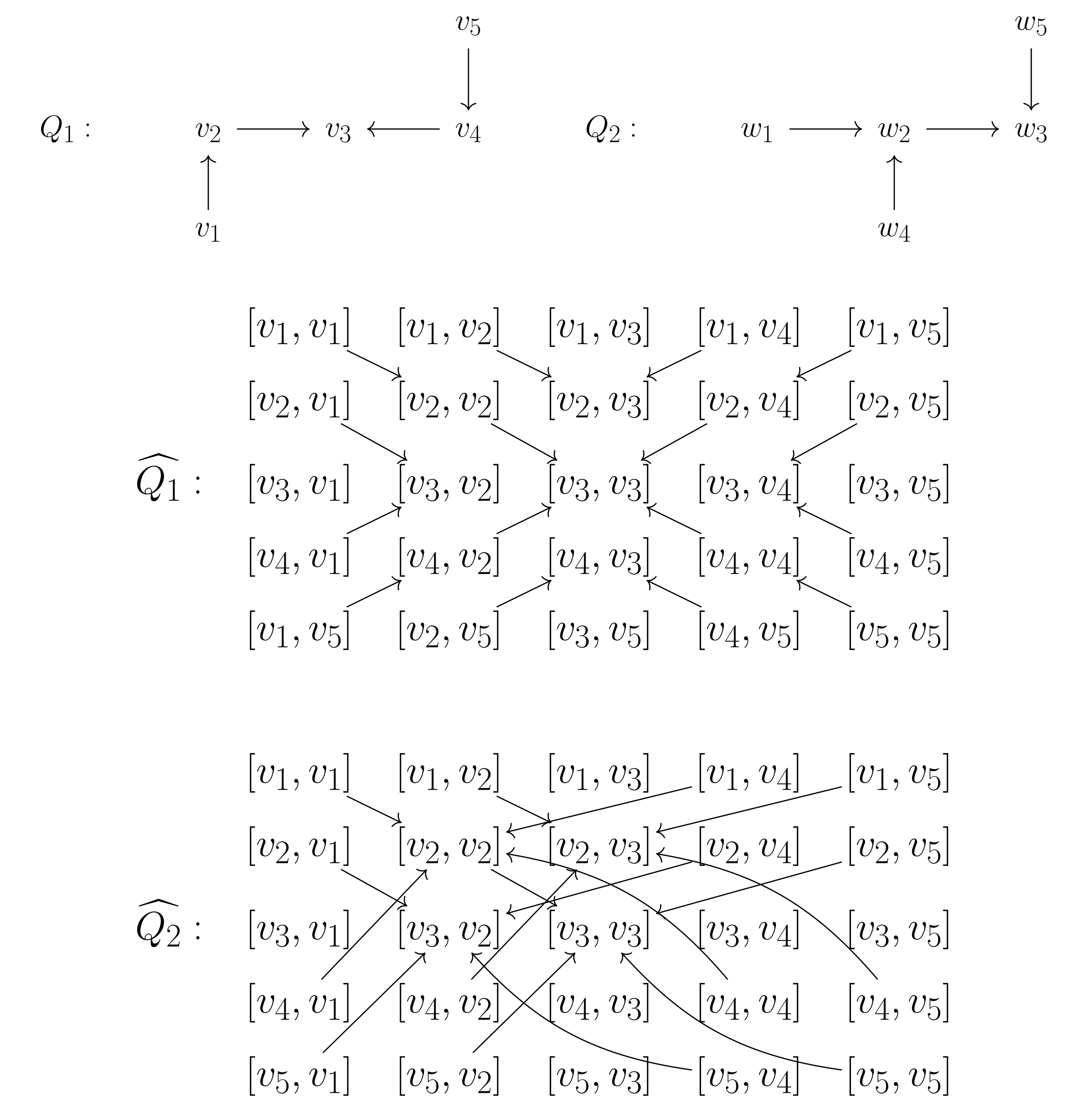
$$\mathrm{GKdim}(L_{\mathbb{K}}(\widehat{Q})) = \begin{cases} 2n-1, & n \text{ is odd} \\ 2n-2, & n \text{ is even} \end{cases}$$

Isomorphisms between Leavitt path algebras

Questions: Suppose Q_1 and Q_2 are two quivers such that

1. If $L_{\mathbb{K}}(Q_1) \cong L_{\mathbb{K}}(Q_2)$, does it imply that $L_{\mathbb{K}}(\widehat{Q_1}) \cong L_{\mathbb{K}}(\widehat{Q_2})$?
2. If $L_{\mathbb{K}}(Q_1) \cong_{gr} L_{\mathbb{K}}(Q_2)$ does it imply that $L_{\mathbb{K}}(\widehat{Q_1}) \cong_{gr} L_{\mathbb{K}}(\widehat{Q_2})$?
3. If $L_{\mathbb{K}}(\widehat{Q_1}) \cong L_{\mathbb{K}}(\widehat{Q_2})$ does this imply that $L_{\mathbb{K}}(Q_1) \cong L_{\mathbb{K}}(Q_2)$?
4. If $L_{\mathbb{K}}(\widehat{Q_1}) \cong_{gr} L_{\mathbb{K}}(\widehat{Q_2})$ does it imply that $L_{\mathbb{K}}(\widehat{Q_1}) \cong_{gr} L_{\mathbb{K}}(\widehat{Q_2})$?

Example



Future Work

1. Investigate how to determine whether a given quiver can be realized as the Kronecker square of another quiver, and in such cases, how to reconstruct a quiver from its Kronecker square.
2. Determine the number of cycles in $\widehat{Q}$ corresponding to given cycles in a quiver Q.

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References

[1] María Guadalupe Corrales García, Dolores Martín Barquero, and Cándido Martín González. On the gauge action of a leavitt path algebra. *Kyoto Journal of Mathematics*, 55(2), June 2015.

[2] Paul M Weichsel. The kronecker product of graphs. *Proceedings of the American Mathematical Society*, 13(1):47–52, 1962.