# Direct Transmission Problem in Several Layers

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## Motivation

Electroencephalography (EEG) is used to measure electrical activity in the brain through the placement of electrodes on the scalp.

- The electrodes provide punctual **values (measurements)** of the electrical signal.
- The electrical activity is **generated** by neurons located inside the brain.

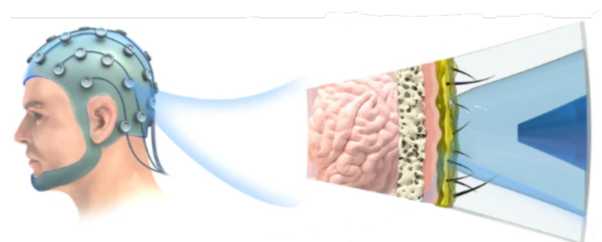


Figure 1. Demonstrating how measurements are made with EEG.

Before being measured, the signal passes through different layers in the head as illustrated in Figure 1.

Three layers on the head will be considered:

- the scalp,
- the skull,
- the brain.

**Questions**:

- Can we recover the full electrical potential inside the head using only the EEG measurements on the scalp?
- Each layer has its own conductivity. If two conductivities are known, can we determine the third one from the measured data?

## Mathematical Modeling of EEG

Inside the brain, neuronal activity creates localized electrical currents. Each active region can be modeled by a current dipole inside the brain domain. Each dipole is defined by the following parameters:

- the **location** $C_q \in \Omega_0$,
- its **moment vector** $\mathbf{p}_q \in \mathbb{R}^3$, indicating the direction and strength of the current.

Restricting the electric potential, $u(x, y, z)$, to the brain domain, the conductivities are given as functions $\sigma_k(x, y, z)$.

- $\sigma_0$ denotes the conductivity in the brain, $\Omega_0$.
- $\sigma_1$ the conductivity in the bone, $\Omega_1$.
- $\sigma_2$ represents the conductivity of the scalp, $\Omega_2$.

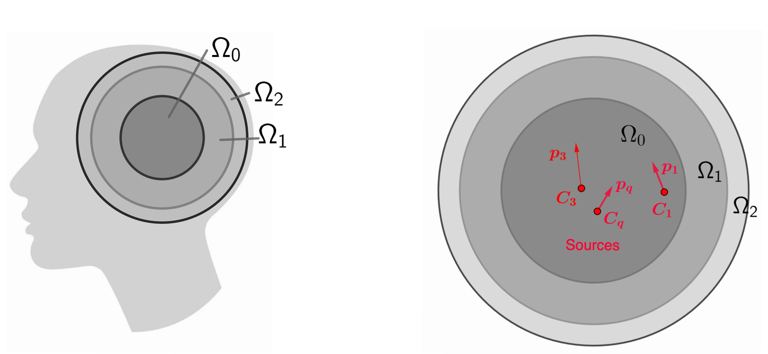



Figure 2. Layered model of the head with associated conductivities and dipoles (in red) represented.

## The Direct Problem

Defining the direct problem as the calculation of electrical potential on the scalp, the following protocol is used:

**Let:**
- $S$ := electric current/potential source in brain
- $\sigma_k$ := conductivity on layer $\Omega_k$
- $r_k$ := outer radius of layer $k$
- $T_k$ := transmission operator at interface $r_k$ (derived in [2])
- $\Delta$ := Laplace operator
- $\Delta^{-1}$ := inverse Laplace operator

**Procedure: DirectProblem**($S, \sigma_k, r_k, T_k$)
- Initialize potentials: $u_0 = \Delta^{-1}(S/\sigma_0)$, $u_1 = 0$, $u_2 = 0$
- **for** $k = 1$ to $2$ **do**
  - Solve Laplace equation in layer $k$: $u_k = \Delta^{-1}(0)$
  - Apply transmission conditions: $T_{k-1}u_{k-1} = u_k$ on $r_{k-1}$
- **end for**
- Apply boundary condition on scalp: $\partial_\nu u_{2|_{r_2}} = 0$
- Evaluate scalp potential: $h = u_2(r_2)$
- **return** $h, u_k(r)$

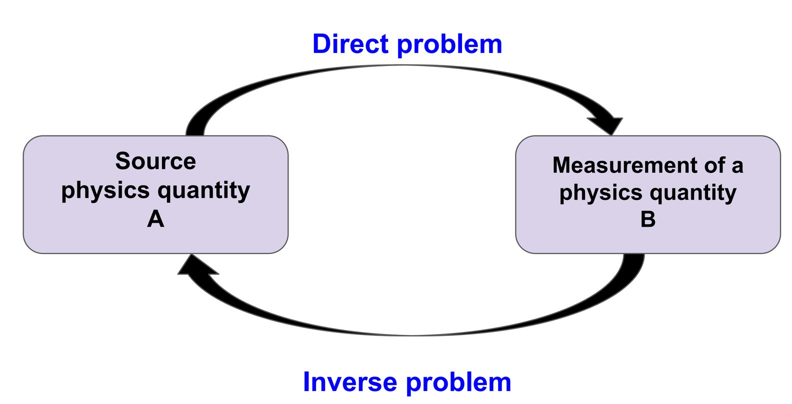
As the electrical signals originate from the brain, the potential on the brain is initialized with respect to the source. In the skull and scalp layers, no electrical potential is produced, thus their respective initializations are set to 0.

Based on the relationships $\sigma_k \Delta u_k = S_k$ and $\sigma_k \partial_\nu u_k = \sigma_{k-1}\partial_\nu u_{k-1}$, the transmissions across the bone and scalp layers are formalized. Lastly, the boundary condition on the scalp is set to 0 since the air holds negligible electrical potential, and the potential on the scalp is the result that is returned.

## From the Direct Problem to the Inverse Problem

In practice, EEG provides the measured scalp potentials, while the conductivities inside the head are unknown. The direct problem computes the potentials from known conductivities, but in applications we face the inverse problem:

**Given the voltages measured on the scalp, can we recover the conductivity inside the skull?**




The direct problem predicts measurements; the inverse problem reconstructs internal parameters.

This setting is an example of a **Calderón inverse problem**: recovering the conductivity of a medium from boundary voltage and current measurements, first posed by Calderón in 1980, see [1].

## Dirichlet-to-Neumann Operator

A key component in Calderón problems, the Dirichlet-to-Neumann (DtN) operator is a map that relates the applied electric potential to the current density across a certain boundary:

$$\Lambda_\sigma : g \mapsto \sigma\, \partial_\nu u_{|\partial\Omega}.$$

## The Inverse Problem

In developing an algorithm for the inverse problem, the following values will be initialized:

- EEG measurements,
- the arc on the scalp where the measurements are taken,
- locations of the sources within the brain,
- number of defined layers.

For simplicity, the conductivity of each layer has been assumed piecewise constant. The algorithm will reconstruct an approximation of the conductivity, $\sigma_1$ or $\sigma_{N-2}$ given that $N \geq 3$.

In a more realistic case, the domain geometry should also be reconsidered to resemble the shape of a head more closely. While this example takes constant conductivity, the non-constant case will also be examined.

Approximation schemes like interpolation will be utilized for the calculation of the conductivities. With this calculation in mind, the following theorem will be explored and verified:

Given $u(x, y, z)$ on $\Omega$ and for any $\varepsilon > 0$, there is a continuous differentiable $\widetilde{\sigma}_{N-2}$ such that

$$|\sigma_{N-2}(x, y, z) - \widetilde{\sigma}_{N-2}(x, y, z)| < \varepsilon$$

and the following conditions are fulfilled:

$$\Delta u_i = 0 \text{ on } \Omega_i \text{ for } i \in \{1, 2\}$$

$$\Delta u_0 = S \text{ on } \Omega_0$$

$$\sigma_2 \partial_\nu u_2 = \sigma_1 \partial_\nu u_1 \text{ on } \partial\Omega_1 \cap \partial\Omega_2 \text{ and } \sigma_1 \partial_\nu u_1 = \sigma_0 \partial_\nu u_0 \text{ on } \partial\Omega_0.$$

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## References

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[2] Ma. Clerc, J. Leblond, J.-P. Marmorat, and C. Papageorgakis. Uniqueness result for an inverse conductivity recovery problem with application to EEG. *Rend. Istit. Mat. Univ. Trieste*, 48:385–406, 2016.

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