Calculating Normal Probabilities without Tables

An Exposition of Bagby (1995)

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Introduction and Motivation

In his 1995 paper, Bagby asks how to compute standard normal probabilities *without using tables*. For a standard normal random variable $X \sim N(0,1)$ and $a > 0$,

$$P(a) = \Pr(0 < X < a) = \frac{1}{\sqrt{2\pi}} \int_0^a e^{-x^2/2}\,dx,$$

an integral with no elementary antiderivative.

Before personal computers were widespread, such probabilities were read from long tables or approximated roughly. Bagby's goal is to construct a **simple, explicit, accurate** approximation $Q(a)$ that

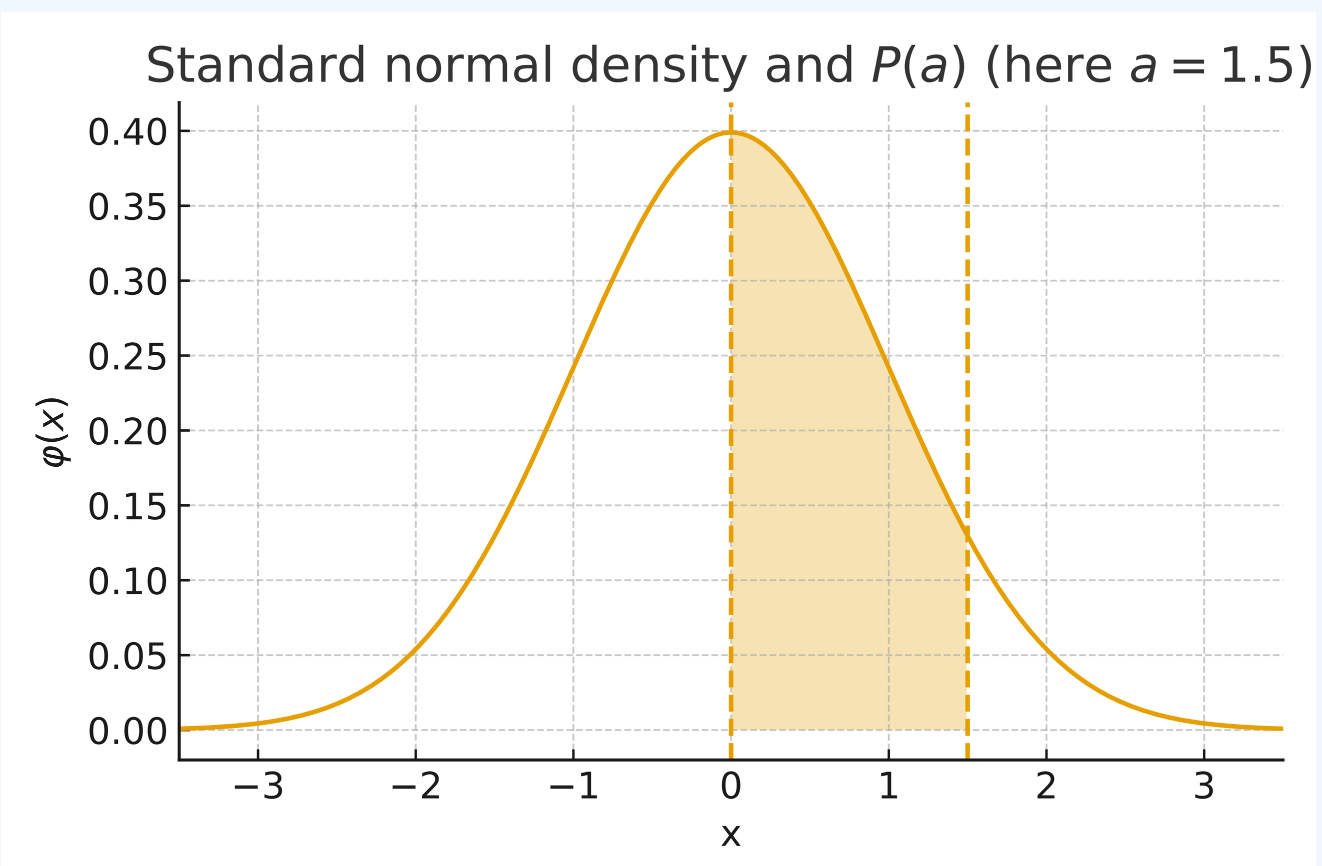
- is easy for hand or calculator computation,
- is very accurate on the range $0 < a \lesssim 3$ used in statistics,
- can replace tables in many practical settings.

The One-Dimensional Integral $P(a)$

The starting point is

$$P(a) = \Pr(0 < X < a) = \int_0^a \varphi(x)\,dx, \qquad \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Geometrically, $P(a)$ is the area under the bell-shaped density between 0 and a.



Area under the standard normal curve giving $P(a)$ (example with $a = 1.5$).

Bagby's strategy is:

- transform this difficult 1D integral using multivariable calculus and numerical analysis;
- then return to an explicit approximation $Q(a)$ for $P(a)$.

Accuracy and Refinements for $P(a)$

Applying the quadrature rule to

$$\int_0^{\pi/4} e^{-\frac{1}{2}a^2 \sec^2\theta}\,d\theta$$

uses only three function values and two endpoint derivatives in θ. For this specific integrand,

$$f(0),\ f\big(\tfrac{\pi}{8}\big),\ f\big(\tfrac{\pi}{4}\big),\ f'(0),\ f'\big(\tfrac{\pi}{4}\big)$$

can all be written in terms of elementary exponentials, giving an explicit approximation $Q(a)$ built from three exponential terms.

Bagby's numerical comparisons show that

$$|Q(a) - P(a)| \lesssim 3 \times 10^{-5} \quad \text{for } 0 < a \lesssim 3,$$

and the error tends to 0 as $a \to 0$ or $a \to \infty$.

Subdividing $[0, \pi/4]$ (for instance into $[0, \pi/8]$ and $[\pi/8, \pi/4]$) can further reduce the error, but the basic three–function-value rule already outperforms standard four-decimal normal tables.

From One Dimension to Two Dimensions

Let X, Y be independent $N(0,1)$ random variables. For $a > 0$,

$$P(a) = \Pr(0 < X < a), \qquad P(a)^2 = \Pr(0 < X < a,\ 0 < Y < a).$$

Using the joint density of (X, Y),

$$P(a)^2 = \frac{1}{2\pi} \int_0^a \int_0^a e^{-(x^2+y^2)/2}\,dy\,dx.$$

Thus the square of a one-dimensional probability becomes a **double integral** over the square $[0,a] \times [0,a]$, with an integrand that depends only on $x^2 + y^2$.

From Product to Double Integral

The continuous identity above mirrors a discrete "sum of sums" picture. Let

$$A = \sum_{i=1}^{n} a_i, \qquad B = \sum_{j=1}^{m} b_j.$$

Then

$$AB = \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{m} b_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j.$$

Riemann sums behave the same way:

$$\sum_i f(x_i)\,\Delta x \quad \text{and} \quad \sum_j f(y_j)\,\Delta y$$

have product

$$\sum_i \sum_j f(x_i) f(y_j)\,\Delta x\,\Delta y,$$

which in the limit gives

$$\int_0^a \int_0^a f(x) f(y)\,dy\,dx.$$

In our case $f(t) = e^{-t^2/2}$, so

$$P(a)^2 = \frac{1}{2\pi} \int_0^a \int_0^a e^{-(x^2+y^2)/2}\,dy\,dx.$$

From Double Integral to a θ-Integral

Because $e^{-(x^2+y^2)/2}$ depends only on $x^2 + y^2$, polar coordinates are natural:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad dx\,dy = r\,dr\,d\theta.$$

On the square $(0,a) \times (0,a)$ in the first quadrant,

$$0 \le \theta \le \frac{\pi}{4}, \qquad 0 \le r \le a\sec\theta.$$

Therefore

$$P(a)^2 = \frac{1}{2\pi} \int_0^{\pi/4} \int_0^{a\sec\theta} e^{-r^2/2}\,r\,dr\,d\theta.$$

Integrating first in r gives

$$\int_0^{a\sec\theta} e^{-r^2/2} r\,dr = 1 - e^{-\frac{1}{2}a^2\sec^2\theta},$$

so

$$P(a)^2 = \frac{1}{\pi} \int_0^{\pi/4} \left(1 - e^{-\frac{1}{2}a^2\sec^2\theta}\right) d\theta = \frac{1}{4} - \frac{1}{\pi} \int_0^{\pi/4} e^{-\frac{1}{2}a^2\sec^2\theta}\,d\theta.$$

The problem is now reduced to approximating a single integral in θ.

Big Picture: A High-Order Quadrature Rule

Bagby's next goal is to build a very accurate rule for

$$\int_a^b f(x)\,dx,$$

and then apply it to

$$\int_0^{\pi/4} e^{-\frac{1}{2}a^2\sec^2\theta}\,d\theta.$$

The resulting rule has the shape

$$\int_a^b f(x)\,dx \approx \frac{b-a}{30}\Big(7f(a) + 16f\big(\tfrac{a+b}{2}\big) + 7f(b)\Big) - \frac{(b-a)^2}{60}\big(f'(b) - f'(a)\big),$$

plus an explicit error term involving $f^{(6)}$.

Main ideas:

- use repeated integration by parts with a carefully chosen polynomial $K(x)$;
- compare a "one-interval" rule and a "two-interval" rule;
- combine them so leading error terms cancel (Richardson extrapolation).

Constructing the Quadrature Rule

On a symmetric interval $[-h, h]$ Bagby chooses a polynomial K with

$$K^{(6)}(x) \equiv 1, \qquad K(h) = K'(h) = K^{(3)}(h) = 0.$$

Integrating by parts several times yields, for smooth f,

$$\int_{-h}^{h} f(x)\,dx = \Big[K^{(5)}f - K^{(4)}f' - K^{(2)}f^{(3)}\Big]_{-h}^{h} + \int_{-h}^{h} K(x)\,f^{(6)}(x)\,dx.$$

A general solution of $K^{(6)} \equiv 1$ is a degree-6 polynomial. Imposing evenness and the boundary conditions at $x = h$ leads to

$$K(x) = \frac{1}{720}(x^2 - h^2)^2(x^2 - 3h^2).$$

After shifting to a general center c (writing $x = c + t$), this produces a refined trapezoid-type rule on $[c-h, c+h]$ with an explicit error term involving $f^{(6)}$.

Error Structure and Final Formula

Applying the construction on $[c-h, c+h]$ (one interval) and on the two halves $[c-h, c]$, $[c, c+h]$ (two intervals) gives two approximations to

$$I = \int_{c-h}^{c+h} f(x)\,dx$$

whose error expansions have different h^4-terms.

Richardson extrapolation chooses a linear combination of these two rules so that the h^4-terms cancel. After simplification Bagby obtains

$$\begin{aligned}\int_{c-h}^{c+h} f(x)\,dx &= \frac{h}{30}\Big[7f(c+h) + 16f(c) + 7f(c-h)\Big] \\ &\quad - \frac{11}{60}h^2\Big[f'(c+h) - f'(c-h)\Big] \\ &\quad + \frac{1}{3600}\int_{-h}^{h}\Big[f^{(6)}(c+x) + f^{(6)}(c-x)\Big]\,x(x-h)^4(5x^2 + 4hx + h^2)\,dx.\end{aligned}$$

The first two lines give the practical rule; the last line is an exact error term. The polynomial weight

$$x(x-h)^4(5x^2 + 4hx + h^2)$$

has a fixed sign on $(0, h)$, so by the mean value theorem

$$\text{Error} = C\,h^7 f^{(6)}(\xi)$$

for some $\xi \in (c-h, c+h)$ and constant $C > 0$.