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Beyond Classical Symmetries: Non-Commutative Structures in the Quantum Realm

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Phase Space

In classical mechanics, systems are described in terms of a space of all possible states of that system, called a phase space.

Phase space is a $2n$-dimensional symplectic manifold (where n is the number of degrees of freedom), coordinatized by canonical position and momentum variables q^i and p_i.

Phase Space of a Pendulum

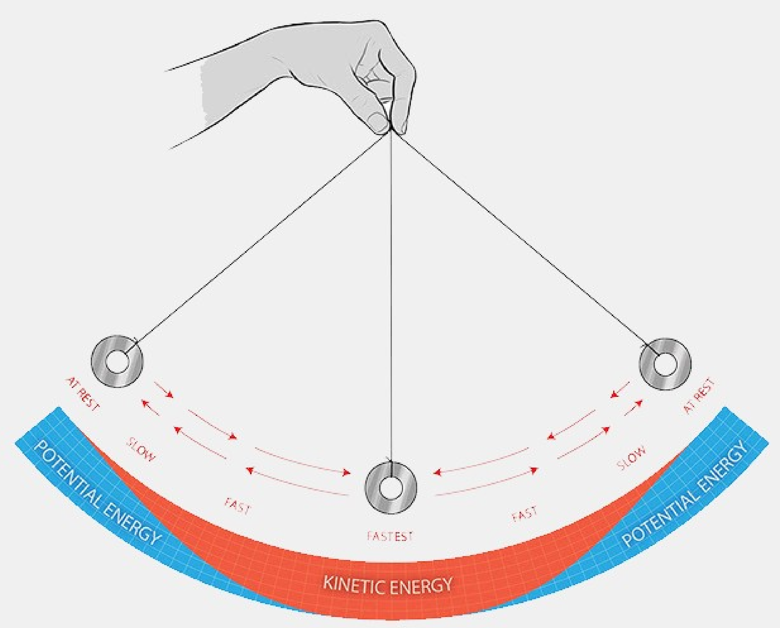

Figure 1.

Given this system, the path through its phase space may look like this:

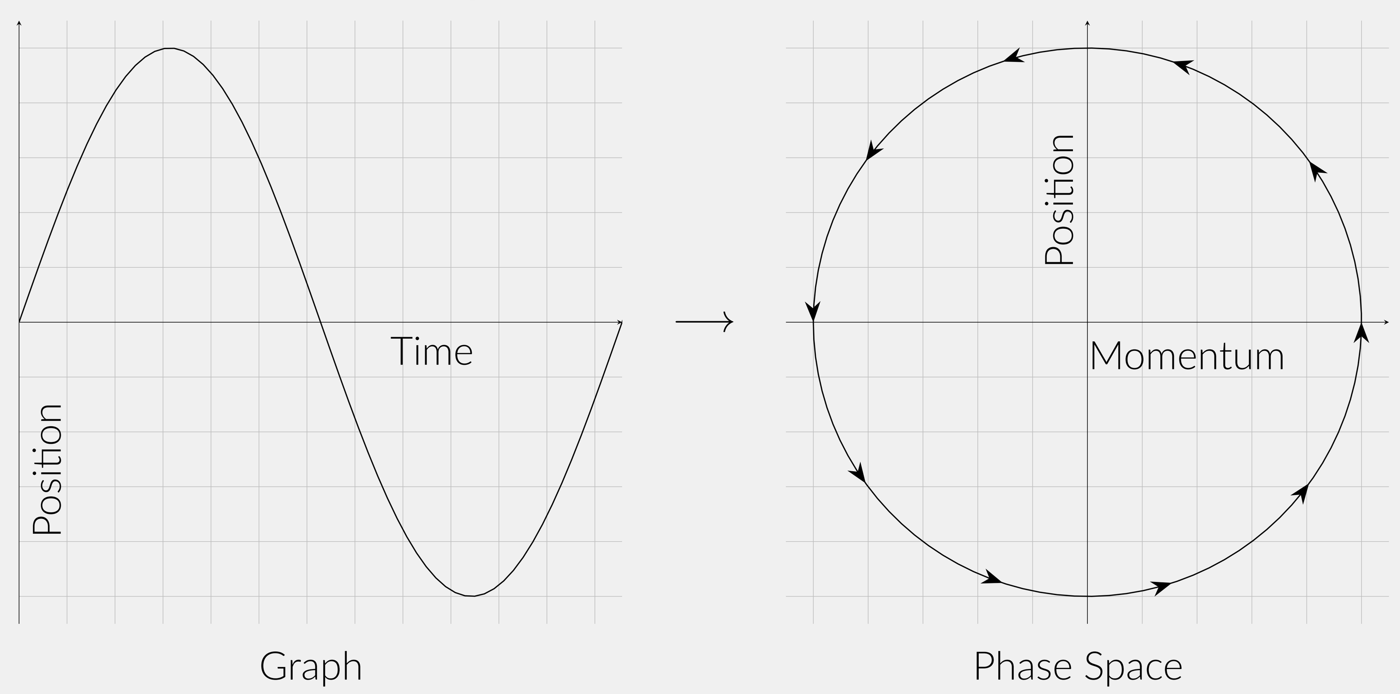

Figure 2.

Poisson Algebra of Observables

Physical observables (like energy, momentum, position) are represented as smooth functions on this phase space M.

The set of all such observables composes a type of function space over $\mathbb{R}$ called a Poisson Algebra, denoted $C^\infty(M)$, with the following properties:

- The product $\cdot$ forms an associative $\mathbb{R}$-algebra
- The product $\{,\}$ (called a Poisson bracket) forms a Lie algebra
- The Poisson bracket is a derivation, meaning for any $x, y, z \in C^\infty(M)$:

$$\{x, y \cdot z\} = \{x, y\} \cdot z + y \cdot \{x, z\}$$

Point-Wise Product of Observables

For two observables $f, g \in C^\infty(M)$, the point-wise product $f \cdot g$ is a new observable in $C^\infty(M)$ defined using the ordinary product of functions:

$$(f \cdot g)(p, q) = f(p, q) \cdot g(p, q)$$

And the point-wise product is commutative, meaning

$$f \cdot g = g \cdot f$$

Poisson Bracket of Observables

For an observable $h \in C^\infty(M)$, we define the vector field generated by h (X_h) and the symplectic form ω by:

$$X_h := \sum_{j=1}^{n} \left(\frac{\partial h}{\partial p_j} \partial_{q^j} - \frac{\partial h}{\partial q^j} \partial_{p_j} \right) \qquad \text{and} \qquad \omega := \sum_{j=1}^{n} dq^j \wedge dp_j$$

For two observables $f, g \in C^\infty(M)$, the Poisson bracket $\{f, g\}$ generates a new observable in $C^\infty(M)$. It is defined as follows:

$$\{f, g\} := \sum_{j=1}^{n} \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q^j} - \frac{\partial f}{\partial q^j} \frac{\partial g}{\partial p_j} = dg(X_f) = X_f(g) = \omega(X_g, X_f)$$

Loosely speaking, $\{f, g\}$ measures the change in f as you "move along" g.

All observables in a phase space can be constructed from the canonical observables $\{p_j\}$ and $\{q^j\}$ using the Poisson bracket and point-wise product.

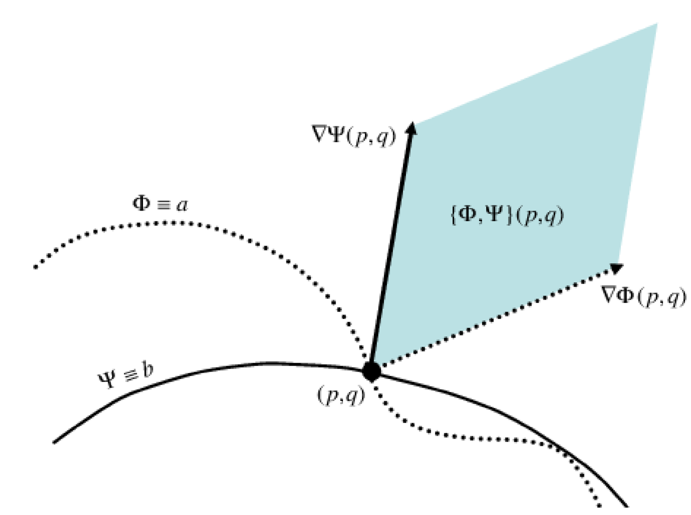

Figure 3.

Observables on the Pendulum

Using the example phase space from Figure 2, some commonly used observables can be defined from the two canonical observables p and q:

$$T(p) = \frac{1}{2}mv^2 = \frac{p^2}{2m} \qquad \text{(Kinetic Energy)}$$

$$V(q) = mg\sqrt{1 - q^2} \qquad \text{(Potential energy)}$$

$$H(p, q) = T(p) + V(q) = \frac{p^2}{2m} + mg\sqrt{1 - q^2} \qquad \text{(Hamiltonian/Total Energy)}$$

$$\frac{df}{dt} = \{f, H\} \qquad \text{(Time Derivative Using the Hamiltonian)}$$

Symmetries of Phase Space

In phase space, symmetries are defined by conserved quantities. A function $f \in C^\infty(M)$ is a conserved quantity iff

$$\{f, H\} = 0$$

Hilbert Space

Analogously to the phase space, Hilbert spaces are used to represent states of a system in Quantum Mechanics.

The Hilbert Space of a quantum system is the complete $\mathbb{C}$-space over all of its observable states (e.g. spin) together with an inner product defined for two state vectors $|\Phi\rangle$ and $|\Psi\rangle$ by:

$$\langle \Phi | \Psi \rangle = \int_{-\infty}^{\infty} \Phi^*(x)\Psi(x)dx \qquad \text{and} \qquad \langle \Phi | \Psi \rangle = \sum_{j=1}^{k} \Phi_j^* \Psi_j$$

for the continuous and discrete cases respectively.

Symmetry in Hilbert Space

In quantum systems, observations change the system, meaning we cannot take measurements in different orders and get the same result. The difference in taking two measurements $\hat{Q}$ and $\hat{P}$ is measured by the commutator:

$$[\hat{Q}, \hat{P}] = \hat{Q}\hat{P} - \hat{P}\hat{Q}$$

An observable represented by the operator $\hat{A}$ is a conserved quantity iff

$$[\hat{A}, \hat{H}] = 0$$

Heisenberg Uncertainty Principle

In Quantum Mechanics, Particles are Waves, and thus do not have simultaneously well defined position and momentum (i.e. wavelength and position). This is given by the Heisenberg Uncertainty Principal:

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

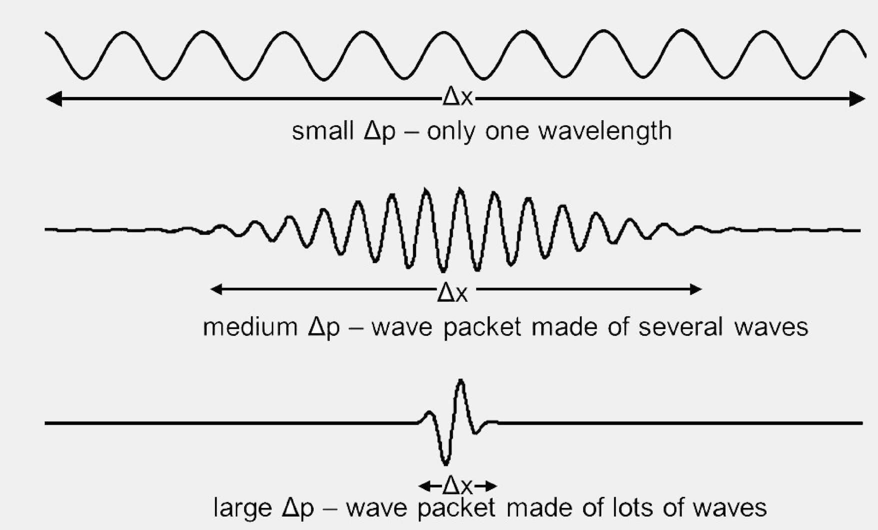

Figure 4.

Non-Commutativity

In the construction of the universal enveloping algebra, we demanded the multiplication of observables to be commutative, however the corresponding quantum algebra does not demand that the underlying objects commute.

$$f \cdot g = g \cdot f \qquad \text{(classical phase space)}$$

$$\hat{Q}\hat{P} \neq \hat{P}\hat{Q} \qquad \text{(quantum Hilbert space)}$$

References

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